Is mathematics a discovery or an invention?

Famous German mathematician Cantor remarks: “The essence of mathematics lies precisely in its freedom.”

Famous German mathematician Kronecker remarks: “God made the integers, all else is the work of man.”

Lexical meaning of “discovery”: to become aware of a fact or situation unexpectedly or during a search.

Lexical meaning of “invention”: to create or design something that has not existed before.

The nature of mathematical realities and how mathematical knowledge is acquired by humans have been a subject of interest in epistemology since Ancient Greek philosophy. In response to this question, two different lines of thought have been put forward, which can be referred to as the exploratory approach and the inventive approach. According to the exploratory approach, mathematical realities are beyond time and space, they are independent of human consciousness, they are objective and absolute and cannot be corrected or altered. The acquisition of mathematical knowledge by man points at a discovery. According to the inventive approach, mathematical knowledge is a product of the intellectual evolution of man, it continues its development in an unceasing process that will never be completed, it is relative and fallible, so it can be reviewed and corrected. The first of these approaches has its roots in Platonism and the latter in Formalism. According to Plato's Theory of Forms, concrete objects in the physical world are irreducible and imprecise equivalents of ideas. They depend on time and space, are temporary and may change. Ideas are on the contrary perfect things that are beyond the sensible world and independent of it, which can only be conceived by thinking. In contrast to Platonism, Formalism asserts that the form outweighs the essence, thus denying that mathematical knowledge is universal.

I regret that I have not come across the mentioned issue until now, although I have been engaged in mathematics for many years. Grasping the paradigm of mathematical thinking obviously requires a special intellectual effort, even for mathematicians. In this paper, I will therefore present an argumentation that can be schematized as in the following to demonstrate the characteristics of mathematical realities and to prove that the exploratory approach explains these characteristics more successfully than the inventive approach. I will also try to answer possible criticisms that can be made by the supporters of the inventive approach. I, however, need to underline that this plea can be made more easily in the primitive periods when the mathematical knowledge seemed to be new.

1. It is an indisputable fact that mathematics is compatible with the universe and suitable for formulating the laws of nature.

2. It is impossible to examine the development of mathematical knowledge chronologically.

3. Mathematical realities are objective, precise and unchangeable. By their nature, they have to be seen as an a priori, and they hence remain independent of human consciousness and sensitivity.

4. Since the exploratory approach either explains these phenomena better than the inventive approach or renders them more expected, it should be preferred that mathematics is a discovery rather than an invention.
Evaluation of 1: The fact that mathematics is suitable for formulating laws of nature is fixed by our experience in physics, chemistry and many other sciences. We accordingly need to investigate which one of the exploratory and inventive approaches explain the compatibility of the mathematics with the universe better? From the outset, the theists present it as an evidence for the existence of God that both of them fail to explain this phenomenon. Nonetheless, we will examine which point of view makes this suitability more expected.

If mathematics was to be said to be an invention of the human mind, would it not be expected that it should be just as compatible with the universe as the chess is that is another invention of the human mind? Which invention from particle accelerators to gene scissors has enabled us to understand the universe as much as mathematics did? On the other hand, the discovery of subatomic particles or DNA allows us to better understand the universe when compared to the invention of particle accelerators or gene shears.

Instead of responding to these questions, the supporters of the inventive approach may try to cast a suspicion on the credibility of this argumentative line by asserting the following claims:

1.1. Since the exploratory approach contends that mathematics is beyond space and therefore independent of the universe, it also fails to make this compatibility expected.

While expressing that it is beyond the universe, the exploratory approach actually aims to say that mathematics is inherent in any universe that can be imagined. The concrete objects that exist in every sensible universe are imperfect counterparts of ideas, even if they are irreducible. Consequently, it is no wonder that laws of nature can be formulated with existing mathematical relations in the world of ideas.

1.2. The mathematical concepts in the world of ideas are abstract by definition and the most prominent feature of abstract structures is that they do not come along in causal relationship. Therefore, the exploratory approach is still insufficient to explain the compatibility of the mathematics with the universe.

Well, we do not deny that mathematical knowledge is product of an abstraction process. On the contrary, after the existence of clues that will enable the discovery, abstraction is the most indispensable phenomenon that plays a very important role in the process of the discovery, whose absence would even prevent the discovery. I will try to concretize this remark in the case of the discovery of the American continent: Of course, the existence of American lands is necessary for the discovery of the American continent. Of course, the compass and ships that are capable of making transatlantic expeditions are needed for the discovery of the American continent. Similarly, in order to acquire mathematical knowledge, an intelligent mind is needed that has the required potential and necessary background knowledge, making the comprehension of mathematical realities possible. Please note that the absence of those does not do harm to the very existence of neither American continent nor the mathematical realities. Christopher Columbus made his mark on the continent, but he has not been honored with the award of giving his name to it, as he could not complete the discovery, because he supposed the lands he set foot on to be Indian lands. Hence, the phenomenon of awareness is indispensable in the process of discovery. Subsequent to becoming aware of the discovery, the first thing
to do is to symbolize the discovered fact or situation with an indicator that has not been used so far. From this point of view, the word “America” is a linguistic element which is formed by the side-by-side combination of vocals and consonants and had not carried any meaning heretofore. Likewise, mathematicians give the name “fractal” to the geometric figures they see on the surface of leaves that display partial resemblance to the whole and proportional refraction. The discovery process is eventually completed by modeling. That the American continent takes its place in the maps or the fractals are pictured nearly perfectly such that they exhibit their characteristic features are each an example of modeling. The geometric shapes on the surfaces of the leaves affect the modeling of fractals as much as the indentations and protrusions on the shores of the American continent affect its illustration on the map. The suitability of mathematics to the universe arises from the modeling process in the discovery.

1.3. Mathematical objects cannot be found in nature. An exact fractal or an exact circle in mathematical sense cannot be pointed at in nature.

The appearance of leaf surfaces may represent a model of a fractal for a mathematician, the appearance of the Moon or the Sun may represent a model of a circle. In addition, nature cannot be limited to physical objects that humans can observe and feel, but it is also connected to many abstract elements that go beyond the senses, nevertheless, can be grasped through thinking.
Evaluation of 2: Before presenting this argument, let us put the emphasis on the difference between mathematical reality and mathematical knowledge that is acquired by man. Since mathematical realities are already out of time, they cannot be studied with a linear understanding of history. The fact that the mathematical knowledge is directly related to the mathematical realities necessitates that it cannot be arranged in a chronological order according to the date on which it was acquired by man, either. After studying his theorem out that had been named in his honor, it is narrated that Pythagoras sacrificed an ox as an expression of gratitude to God. However, the mathematical reality called “The Pythagorean Theorem” was also known by the Babylonians 1000 years ago, and it was proved by Euclid about 200 years after Pythagoras. When did then the Pythagorean Theorem gain accuracy? During the period of the Babylonians, in the age when Pythagoras lived or when it was proved by Euclid? Where should the mathematical reality called “The Pythagorean Theorem” be placed in the chronological history? Or was it true, even though we did not know that it was true, was it in other words independent of human consciousness and of time? If mathematics had been an invention, then mathematical findings could have been placed in a chronological order, just like the other means of inventions such as transport vehicles. As a result, mathematics is not only out of space, but also out of time.

Despite the given example, the supporters of the inventive approach may tend to argue that the development of mathematical knowledge can nonetheless be examined in a chronological line and to strengthen this view with the following arguments:

2.1. Mathematical knowledge of mankind has developed in a process which depends on the biological evolution of man. It is obvious that animals also have the ability to comprehend mathematical relations. For instance, estimating a certain quantity and acting afterwards according to it are among the capabilities of animals. This even directs very basic decisions of predators such as hunting their potential gregarious victims or giving up.

I only share the conviction that animals have the ability to partially comprehend mathematical relations. They, however, cannot be expected to make high-level mathematical inferences because of their limited intelligence. There is a huge difference between the complexity of mathematical inferences of human beings and of other primates that are closely related to our species according to the evolution theory. Thus, human beings’ abstraction and modeling ability cannot be reduced to the biological evolution process. Making high level mathematical inferences is only possible with a mind capable of distinguishing between the notions of accuracy and inaccuracy and establishing cause-effect relationships, with a will capable of teleological tendencies, and with a consciousness capable of creating awareness and implementing intentionality. Continuing to reduce these special characteristics of human beings to the biological evolution process is nothing but stubbornness.

2.2. The development of the mathematical knowledge throughout the history has always depended on the cultural evolution of societies. From the findings of archaeological excavations, one deduced that mathematics has already been utilized by Egyptians in 3000 BC. Accordingly, two different views were put forward: According to the first, the landowners in Ancient Egypt had to pay taxes in proportion to the acreage of the land they had, and for that reason, submerged areas drown by the flood of the Nile had to be discounted. According to the latter, the priests that formed the bourgeois class of this period
produced games based on arithmetic to kill time and the basis of mathematics was thus established. Both views uphold a common fact that the birth of mathematics and the cultural structure of the society were closely related.

First of all, we need to distinguish between pure and applied mathematics. Whether or not the priests use mathematics to eliminate their boredom or the landowners use it to fulfill their economic needs does not affect the nature of pure mathematical realities. What is said here is as absurd as that astronomy was once subservient to astrology which was however ridiculously true. In mathematics, integral is only a calculation method; it is not a force majeure for calculating the volume of overflowing water from the Nile. It is the physicists' responsibility to place the time on the x-axis. Biology has only expanded the field of application of chemistry, and it is obvious that chemistry could actually exist without vitality, too. Physics only constitutes an applicable field of mathematics, but the validity of mathematical realities is not dependant on whether or not they are applied.

Nevertheless, if the development of mathematical knowledge is related to the cultural evolution of societies, would it not be possible to determine which of these allegations is correct by following this process backwards? We have a large-scale knowledge about the process of the biological evolution of our species that dates back to millennials. On the other hand, these allegations indicate that the first findings on mathematics belong to completely dissimilar sub-disciplines of either geometry or arithmetic, which can be clearly separated from each other. If the answer to the previous question is that there is no such necessity despite all this, we can argue that the incompatibility of these two claims will allow the ascertainment that the mathematical development is not related to the cultural structure. For example, one can ask how the peasants and priests, who are in completely different social classes, have come up with the same pursuit in the very first years of mathematics. Because any of these allegations cannot be refuted directly, one may suppose that both materials regarding tax treatments and materials used by clergy have been found in archaeological excavations. If this was not the case either, the legitimacy of the argument you present is questionable, it does not go beyond just being a guess. Furthermore, the accuracy of the first one of these allegations would be a golden opportunity for Platonists to reinforce their point of view and the accuracy of the latter one would be similarly a golden opportunity for Formalists, which evokes a feeling that your argument is based on fiction.

2.3. The development of the mathematical knowledge of mankind throughout the history is a gradual progress. Mathematicians have been developing new methods from very different perspectives to solve a particular problem, but this has never been something that one can refer to as a revolution that radically changes the understanding of the nature of mathematical knowledge. Mathematical knowledge has mostly developed in a natural and usual process, which is far away from being surprising. To illustrate this fact, one can recall the renowned German mathematician Hilbert who was able to identify 23 mathematical problems that he predicts to be solved in the next century, suggesting that their solutions will revive the disciplines they are concerned with. A process with such characteristics is typical in the development of inventions, but in the realization of discoveries such a process is not expected. For the very same reason, it is not uncommon for a mathematical finding to be revealed simultaneously by two unwitting mathematicians living on two separate ends of the world.
Regarding the previous question, the fact that two mathematicians of different educational, political, religious and socio-economic backgrounds can do the same work at the same time reveals that the development of mathematical knowledge is not about the cultural structure of societies.

I do not agree with the assertion that mathematics has progressed gradually throughout the history. The mathematical reality that the equation $a^n + b^n = c^n$ has no solutions in integers for $n \geq 3$ was postulated by French jurist Fermat in 1637 and could first be proved by British mathematician Andrew Wiles in 1994. Although Fermat stated for speculative purposes that he had found a great proof of the theorem, but there was not enough room on the paper to write it, Andrew Wiles had used techniques that were impossible to know at that time. Consequently, becoming aware of mathematical realities does not require waiting for the result of a gradual process of progress. That is why a bunch of other mathematical findings relying on the Fermat’s Postulate could be considered to be correct, even before it was proved, which guarantees the constant development of mathematical knowledge of man.

Nevertheless, suppose that mathematical knowledge has progressed gradually, and that it has developed in a natural process, which is often far away from being surprising. This is due to the directing of the tools that will make the discovery possible, e.g. the current inadequacy of the knowledge the human beings acquired. That Christopher Columbus’s compass directed him to the American continent or that his ship had only the quality necessary to travel to the American continent provides a reasonable basis for the earlier discovery of the American continent in comparison to the Australian continent. But this does not mean that the American continent has existed there before the Australian continent has. This is why two mathematicians, who are unaware of each other, are likely to do the same work at the same time.

The fact that only ten of Hilbert’s problems could have been solved so far demonstrates that his intentions were in vain. Although these problems have become the center of attention of the mathematicians from all over the world and all the effort has been spent on them thanks to Hilbert, more than half of them have remained unsolved over the 119 years going on after Hilbert announced them, which proves that the development of mathematical knowledge is not predictable. Furthermore, contrary to what you think, Gödel’s negative answer to Hilbert’s question of whether mathematics is complete and consistent or not has fundamentally altered the nature of mathematical knowledge. It even will be used as an argument against the exploratory approach in the following.
Evaluation of 3: Let us directly move to possible criticisms against that mathematical reality is objectively and precisely correct, unchangeable and uncorrectable, independent of human consciousness and sensibility, as supporters of the inventive approach will antagonize from the very beginning and radically.

3.1. Gödel's Incompleteness Theorems state that there always is a proposition whose accuracy or inaccuracy cannot be determined with a finite number of axioms, and that an axiomatic system cannot prove its own consistency. How can a structure that exists in the world of perfect ideas not have these features? The only way out is to accept that mathematics is a product of the human mind built on generally accepted axioms.

Here, the concepts of mathematical reality and mathematical knowledge should again be distinguished. Gödel's Incompleteness Theorems are not interested in the nature of pure mathematical realities or how they exist. They only comment on how humans grasp the mathematical realities and how they acquire mathematical knowledge. We, anyway, do not reduce man's acquisition of mathematical knowledge to the discovery of mathematical realities with the help of his intuitions, but he also tries to confirm them with evidence. For this reason, he needs axioms that are accepted by everyone, self-evident, and therefore are the forefront of other propositions. Of course, a solid building can only be built on a suitable ground. Gödel's Incompleteness Theorems display from this point of view the limits of known axiomatic systems. There is no ground which can guarantee that a building built on it will not collapse in any earthquake that may occur. So, Gödel's Incompleteness Theorems do not take a philosophical approach to mathematical realities, but only clarifies the process of obtaining mathematical knowledge by man. Let us recall the fact that the obscurity of mathematical realities does not play a role in their disposition, but they maintain their existence independently of human consciousness.

I do not want to fall into a genetic fallacy, but after finding the theorems named after him, Gödel was among the famous representatives of mathematical Platonism. He argued that mathematical intuition is as reliable as physical senses. As a matter of fact, many mathematicians in the history expressed that they experienced the feeling of coming across a mathematical reality thanks to their intuitive senses that led to mathematical discovery.

3.2. Mathematical knowledge is relative and fallible. To illustrate, Frege, a renowned German mathematician who is considered to be one of the founders of modern logic, tried to build the arithmetic on a more solid basis by founding the Fregean sets theory presented in his work called "The Foundations of Arithmetic". Bertrand Russell, however, showed that Frege's theory was inconsistent due to the paradox named after him. In order to eliminate this paradox, Bertrand Russell then defined new axioms and consequently proposed the theory of types. Accordingly, mathematical information can be reviewed and corrected.

If Frege had been aware of Gödel's Incompleteness Theorems at the time he was living, he would not have tried to build arithmetic on precisely solid basis, being convinced of the impossibility of such a concept. He would have known that no matter how sagaciously he had determined his axioms, there always would have been a proposition whose accuracy or inaccuracy he could not have been able to prove. As Russell put forward such a proposition, Frege would not have been so upset because of the dissolution of the foundations of his work and would not have had such a disappointment. Here, Frege's
attitude is wrong. The contradictions that have been noticed must be eliminated by a few minor changes to the axioms, as Russell did. However, this does not necessitate the conclusion that mathematical realities can be reviewed and corrected, as these are only impediments that are related to the consideration of a chronological development of mathematical knowledge, which furthermore arise during the process of the acquisition of mathematical knowledge by humans, not at the stage of the discovery of mathematical realities, but at the stage of their confirmation through proofs. As discussed in the second argument, a chronological examination of the development of mathematical knowledge is however unreasonable. The validity of mathematical realities should rather be considered to be independent of time. Today, Russell’s Paradox means nothing to mathematicians, because there is not such a paradox in the theory of types. One can deduce that mathematical knowledge can be reviewed and corrected, if and only if he or she takes the transition from the Fregean sets theory to Russell’s theory of types into consideration. However, the axioms that shape Frege’s theory are today declared null and void. It is therefore not prejudicial to assume that Fregean sets theory and Russell’s paradox have never existed at any time in the course of history. Similarly, we can respond to any inconsistencies in the future by going one step beyond the time when this inconsistency is detected. Since this discrepancy is no longer meaningful to mathematicians thanks to the new theory established after that day, it can well be assumed that such a discrepancy has never been encountered in history as well. From this point of view, one can conclude that these inconsistencies are not inherent in the nature of mathematical realities that are considered to be independent of time, to be primordial and eternal. Assuming that the time will flow forever, it will be possible to go beyond all inconsistencies.

3.3. With a trite expression, you say that mathematics is independent of human consciousness, that if we encountered an unknown form of life in a remote corner of space, the common language would be mathematics. If a living species other than man had had the ability to reason, the mathematics they would develop would however have been much different. For instance, if a bird species that can fly had been able to make logical inferences, it would definitely start with the three-dimensional space geometry, instead of with the two-dimensional plane geometry.

Yes, this is something impossible to deny. But this argument only proves that our mathematics and the mathematics that arises in the aforementioned situation will not cover each other. But, there surely will be a lot of overlaps. As emphasized in the earlier chapters of this paper, animals are capable of estimating a certain quantity and acting according to it, and in such a case, counting numbers would therefore take place in the mathematics of birds as well, albeit with different indicators, and such fundamentals would then form the common points of both mathematics. We have no objections to such a logical deduction. But if you tend to say that one of these two mathematics should be invalid, then we need to change our attitude, as these two mathematics do not maintain their validity in spite of each other. Contrarily, both are accurate in a way that they do not indicate any inconsistency with each other. In the words of Hardy, a British mathematician, “317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but because it is so, because mathematical reality is built that way.”
Evaluation of 4: Since the exploratory approach better explains the compatibility of the mathematics with the universe and its appropriateness to formulate the laws of nature, since the possibility of examining the development of mathematical knowledge with a chronological understanding of history that actually should have been promoted by the inventive approach is unreasonable, since one can successfully respond to possible criticisms of the supporters of the inventive approach that attack the very fundamental characteristics of mathematical realities based on the exploratory point of view and the reverse of that is impossible (in 3.2, it has been described that mathematical knowledge can be assumed not to be relative, fallible, modifiable or correctable), it is rationally preferable that mathematics is a discovery rather than an invention.

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